Analytical Determination of the Adjoint Vector for Optimum Space Trajectories

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This paper describes a method of obtaining an analytic approximation for the initial adjoint vector on minimum-propellant space trajectories. The adjoint vector is obtained from a series expansion about the optimal impulsive trajectory, satisfying the desired boundary conditions, which are assumed known. First- and second-order corrections to the impulsive adjoint vector are calculated by solving a system of linear algebraic equations requiring very little computing time. Using the resulting corrections to the adjoint vector results in greatly reduced convergence time for fixed-thrust trajectories. The method also provides an analytic estimate of total burn time which can be used for mission performance computation.

Nomenclature

a = upper bound on thrust-acceleration
a_t = thrust-acceleration
g = gravitational acceleration
G = gravity-gradient matrix (symmetric)
h = inverse of p
I = identity matrix
p = magnitude of the primer vector
r = position vector
t = time
t_f = final time
t_k = time of an impulse
Δt_k = shift in the centroid of a burn
u = control (6-vector)
v = velocity vector
ΔV = impulse velocity
x = state (6-vector)
ψ = adjoint (6-vector)
λ = primer vector
Φ = state transition matrix
τ_k = burn time for the kth burn

Introduction

THE application of optimal control theory to the problem of fuel optimal space trajectories inevitably results in a two-point, nonlinear boundary-value problem. Such problems can be solved, in general, only by an iterative technique. The most common of these techniques, the "shooting method," requires that the unknown initial conditions (adjoint variables) be guessed and the equations of motion integrated forward. Differential corrections are then applied until the final boundary conditions are satisfied. Since the final conditions are highly sensitive to errors in the initial adjoint variables, this technique is greatly enhanced by accurate guesses. Inaccurate guesses lead to excessive computing time and/or divergence of the iteration process.

In this paper, a method is developed for generating more accurate estimates of the initial adjoint vector. This method is based on the idea that the impulsive trajectory represents the limit of the fixed-thrust case as the thrust level is increased to infinity. The optimal control laws for the impulsive case were first derived by Lawden[1] using such a limiting argument. The sense in which the limit is approached was made mathematically precise by Neustadt.[2] Pines[3] first suggested using the impulsive adjoint vector as an initial guess and this idea was successfully implemented by Handelsman.[4]

The method developed here carries this approach one step further. Instead of just using the impulsive adjoint vector as first guess for the finite-thrust adjoint vector, a solution for the finite-thrust adjoint is sought in the form of a series in $1/a$, where a is the maximum thrust-acceleration. The "zeroth" order terms in this expansion are the impulsive adjoint. Clearly, this technique is most accurate for high thrust levels since the expansion is about the impulsive case. However, limited numerical experience to date has indicated that the approximation works well into the range of thrust levels appropriate to electric propulsion.

The analysis which follows assumes a constant thrust-acceleration. This assumption is made mainly to simplify the presentation. The case of constant thrust requires expansion in two independent variables (e.g. initial thrust-acceleration and exhaust velocity) and will be presented in a subsequent paper. In the following analysis, it is assumed that the optimal impulsive trajectory and its adjoint are known; calculation of such trajectories is rapid using modern techniques.

Equations of Motion and Necessary Conditions

The equations of spacecraft motion can be written

$$\dot{x} = f(x) + u \tag{1}$$

where the 6-vectors

$$x = \begin{pmatrix} r \\ v \end{pmatrix} \quad u = \begin{pmatrix} 0 \\ a_t \end{pmatrix} \quad f = \begin{pmatrix} v \\ g(r) \end{pmatrix}$$

are conveniently partitioned into 3-vectors as shown: r de-

Presented as Paper 69-916 at the AIAA/AAS Astrodynamics Conference, Princeton, N. J., August 20–22, 1969; submitted August 26, 1969; revision received June 17, 1970. The analytical methods of trajectory analysis research is supported by the Office of Space Science and Applications, NASA Headquarters, under NASA Grant NGR 31-001-152. J. W. Haughey is the NASA Program Monitor, and W. E. Miner, NASA Electronic Research Center, is Technical Monitor. This work made use of computer facilities supported in part by the National Science Foundation Grants NSF-GJ-34 and NSF-GU-3157.

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notes position, v denotes velocity, a_t denotes thrust-acceleration, and g denotes gravitational-acceleration. For present purposes, g will be taken to represent an inverse square field, although this is by no means necessary.

The adjoint equations corresponding to Eq. (1) can be written

$$\dot{\psi} = -F^T\psi \tag{2}$$

where $F = \partial f/\partial x$. This 6×6 matrix is partitioned into 3×3 blocks

$$F = \begin{pmatrix} 0 & I \\ G & 0 \end{pmatrix}$$

where G is the "gravity gradient" matrix $\partial g/\partial r$. Similarly, it is also convenient to partition ψ into two 3-vectors

$$\psi = \begin{pmatrix} -\dot{\lambda} \\ \lambda \end{pmatrix}$$

where λ, the adjoint vector associated with velocity, is known as the "primer vector."[1]

In this paper we consider the case of bounded thrust-acceleration, $0 \leq |a_t| \leq a$, with fixed initial and final conditions, $x(0) = x_0$ and $x(t_f) = x_f$, and given flight time t_f. The fuel optimum control $a_t(t)$ is that control which transfers the system defined by Eq. (1) from x_0 to x_f in the prescribed time t_f and minimizes $J = \int_0^{t_f} |a_t| dt$; J is known as the "characteristic velocity" and is a direct measure of propellant consumption.

Necessary conditions which must hold on the optimal trajectory are stated in terms of the primer vector[1]

$$\begin{aligned} a_t &= a\lambda/p \quad \text{when} \quad p > 1 \\ a_t &= 0 \quad\quad\ \ \text{when} \quad p < 1 \end{aligned} \tag{3}$$

where $p = |\lambda|$, the magnitude of the primer vector. Eqs. (3) express the well-known result that (excluding the special case of singular arcs) the optimal control alternates between periods of maximum thrust (when $p > 1$) and no thrust (when $p < 1$) and the optimum thrust direction is always aligned with the primer vector. The quantity $p - 1$ is called the "switching function." Figure 1 shows a plot of p vs t, together with the corresponding optimal thrust level. During thrusting periods Eq. (3) can also be written

$$u(t) = \frac{a}{p(t)} M\psi(t) \quad M = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \tag{4}$$

where M is a 6×6 matrix.

In the case where the upper bound on thrust acceleration becomes infinite ($a \to \infty$), the optimal control becomes a sequence of velocity impulses

$$\lim_{a\to\infty} a_t(t) = \sum_k \Delta\bar{V}_k \delta(t - t_k) \tag{5}$$

where δ indicates the Dirac delta and t_k is the kth "impulse point."

Necessary conditions for an optimal impulsive trajectory can also be stated in terms of the primer vector[5]: at impulse points $p = 1$ and the impulse is applied in the direction of λ; at all other points $p < 1$. Therefore Eq. (5) can be written

$$\operatorname{Lim}_{a\to\infty} a_t(t) = \sum_k \Delta V_k \lambda_k \delta(t - t_k) \tag{6}$$

These conditions are analogous to the finite-thrust case.

Expansion about the Impulsive Trajectory

The analysis which follows is based upon the assumption that an optimal impulsive trajectory is the limit of a sequence of optimal finite-thrust trajectories as $a \to \infty$.

Once the initial and final conditions (x_0 and x_f) and the flight time (t_f) are prescribed, it is assumed that the optimal impulsive trajectory satisfying those conditions is known.[7] One approach for calculating multi-impulse optimal trajectories is outlined in Ref. 5; using the program based on these techniques[6] calculation of such trajectories requires about 5 sec of IBM 360/65 computer time.

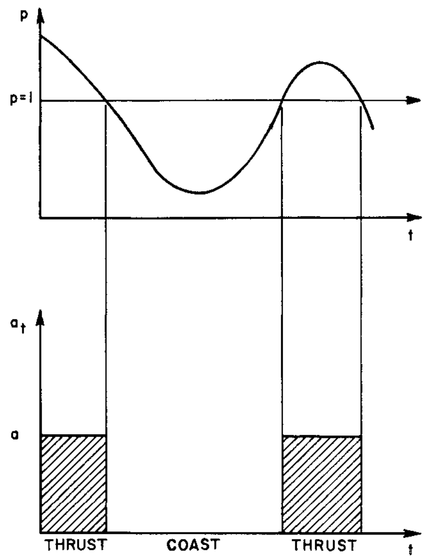

Fig. 1 Optimum finite-thrust primer and thrust history.

In the following analysis, the state and adjoint vectors on the optimal impulsive trajectory are denoted by $x^{(0)}$ and $\psi^{(0)}$, respectively. It is now desired to determine the optimal finite-thrust trajectory satisfying the same boundary conditions. We assume that the state and adjoint vectors for the finite-thrust case can be expanded about the impulsive case as series in $1/a$, as follows:

$$\begin{aligned} x &= x^{(0)} + x^{(1)}/a + x^{(2)}/a^2 \ldots \\ \psi &= \psi^{(0)} + \psi^{(1)}/a + \psi^{(2)}/a^2 \ldots \end{aligned} \tag{7}$$

Using the differential Eqs. (1) and (2), recursion relationships are sought in order to calculate the higher order corrections. Most of the difficulty arises from the additional requirement that the optimality conditions (3) be satisfied to each order. For this reason, it is also necessary to expand the thrust intervals on the finite-thrust trajectory as series in $1/a$, as follows:

$$\tau_k = \tau_k^{(1)}/a + \tau_k^{(2)}/a^2 + \tau_k^{(3)}/a^3 + \ldots \tag{8}$$

where τ_k is the length of the kth thrust interval. Nore that as $a \to \infty$, $\tau_k \to 0$. Subsequent analysis shows that $\tau_k^{(1)} = \Delta V_k$, the kth velocity impulse. Therefore, as $a \to \infty$, we have $a\tau_k \to \Delta V_k$.

Define $\Delta x = x - x^{(0)}$, $\Delta\psi = \psi - \psi^{(0)}$, and $\Delta u = u - u^{(0)}$. Proceeding formally, the differential equations satisfied by these differences are (including terms up to second order)

$$\Delta\dot{x} = F\Delta x + \Delta u + q + \ldots, \quad \Delta\dot{\psi} = -F^T\Delta\psi \tag{9}$$

where

$$q = (1/a)\Delta x^T(\partial^2 f/\partial x^2)\Delta x = \tfrac{1}{2}\Delta r^T(\partial^2/\partial r^2)\Delta r$$

It can be shown that Δr is always of order $1/a$, and therefore q is of order $1/a^2$.[7]

The solution to the first of Eqs. (9) is

$$\Delta x(t) = \int_0^t \Phi(t,\xi)[\Delta u(\xi) + q(\xi)]d\xi \tag{10}$$

where $\Delta x(0) = 0$, since both impulsive and finite-thrust trajectories satisfy the same initial conditions. Φ is the state transition matrix evaluated along the impulsive trajectory,

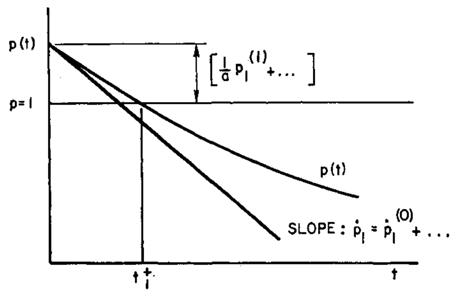

Fig. 2 Geometry of an initial burn.

$\Phi(t,\xi) = \partial x(t)/\partial x(\xi)$ expressing the adjoint solution in terms of Φ,

$$\Delta\psi(t) = \Phi^T(\xi,t)\Delta\psi(\xi) \tag{11}$$

The difficulty in truncating the series (7) is in the evaluation of Δu. Since $u^{(0)}$ is a series of impulses, Δu can never be "small;" however, from Eq. (10) it is seen that Δu appears under the integral; the integral of Δu is indeed small for all a.

Evaluating Eq. (10) at $t = t_f$, and imposing the condition that both finite-thrust and impulsive trajectories satisfy the same final conditions, namely $\Delta x(t_f) = 0$,

$$\int_0^{t_f} \Phi(t_f,\xi)[u(\xi) - u^{(0)}(\xi) + q(\xi)]d\delta = 0 \tag{12}$$

Consider the first two terms under the integral during the kth thrust period τ_k. From Eqs. (4) and (6)

$$u(\xi) = [a/p(\xi)]M\psi(\xi) \tag{13}$$

$$u^{(0)}(\xi) = \Delta V_k M\psi^{(0)}(\xi)\delta(\xi - t_k) \tag{14}$$

Before proceeding it is necessary to expand Φ. A property of the transition matrix is

$$\Phi(t_f,\xi) = \Phi(t_f,t_k)\Phi(t_k,\xi) \tag{15}$$

Assuming τ_k is small (which is true as $a \to \infty$), then $(\xi - t_k)$ is small and $\Phi(t_k,\xi)$ can be expanded in a Taylor series as follows:

$$\Phi(t_k,\xi) = I - (\xi - t_k)F + [(\xi - t_k)^2/2]R + \ldots \tag{16}$$

Similarly Eq. (7) implies that the primer magnitude can be expanded as follows:

$$p(t) = p^{(0)} + p^{(1)}/a + p^{(2)}/a^2 + \ldots \tag{17}$$

We denote the term $1/p$ which appears in Eq. (13) by h. Then, as shown in Appendix A, it is possible to expand $h(\xi)$ as follows:

$$h(\xi) = h_k + \dot{h}_k(\xi - t_k) + \ddot{h}_k[(\xi - t_k)^2/2] + \ldots \tag{18}$$

where h, $\dot{h}$, and $\ddot{h}$ are series in $1/a$ and are functions of $p_k^{(i)}$ (the primer magnitude evaluated at the kth impulse point). These series are contained in Appendix A.

Using Eqs. (11–18), and integrating over the kth thrust period, leads to the following results:

$$\int_{\tau_k} \Phi(t_f,\xi)u^{(0)}(\delta)d\xi = \Delta V_k \Phi(t_f,t_k)M\Phi^T(t_f,t_k)\psi_f^{(0)} \tag{19}$$

Also,

$$\int_{\tau_k} \Phi(t_f,\xi)u(\xi)d\xi = a\int_{\tau_k} h(\xi)\Phi(t_f,\xi)M\Phi^T(t_f,\xi)d\xi\psi_f = \Phi(t_f,t_k)I_k\Phi^T(t_f,t_k)\psi_f \tag{20}$$

where I_k denotes the symmetric matrix

$$I_k = a\int_{\tau_k} [Mh_k + (M\dot{h}_k - Nh_k)(\xi - t_k) + \tfrac{1}{2}(M\ddot{h}_k - N\dot{h}_k + Qh_k)(\xi - t_k)^2]d\xi \tag{21}$$

and

$$N = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad Q = \begin{pmatrix} I & 0 \\ 0 & G \end{pmatrix}$$

In the above equations $\psi_f^{(0)}$ and ψ_f denote the final values of the adjoint vector for the impulsive and finite-thrust cases, respectively.

Before we can evaluate I_k, it is necessary to determine the limits of integration. This is done in the next section.

Switch Point Analysis

The limits of integration for I_k are the switch times t_k^- and t_k^+. These are evaluated by enforcing Eq. (3) through each order in $1/a$. (An alternative way of deriving the same conditions is to require that the Hamiltonian is constant through each order.) One condition is derived at each switch point for each order. Because of the substantial differences in the behavior of the primer on terminal and on interior burns, these cases are treated separately.

It has proved convenient to use the burn duration $\tau_k = t_k^+ - t_k^-$ and the centroid $t_{ck} = \frac{1}{2}(t_k^+ + t_k^-)$ of the thrust interval as parameters instead of t_k^- and t_k^+. The behavior of the primer magnitude for a terminal burn is shown in Fig. 2. For the impulsive case, obviously $t_{ck} = t_k$. For finite thrust, however, t_{ck} deviates from t_k; the deviation is denoted by $\Delta t_k = t_{ck} - t_k$. Since $\Delta t_k \to 0$ as $a \to \infty$, it seems reasonable to expand Δt_k in the form

$$\Delta t_k = \Delta t_k^{(1)}/a + \Delta t_k^{(2)}/a^2 + \ldots \tag{22}$$

For a terminal burn, the centroid is displaced exactly half the duration (since t_k is at one end). Comparing Eq. (22) to Eq. (8) gives for the initial and final burns

$$\Delta t_1^{(i)} = \tfrac{1}{2}\tau_1^{(i)} \qquad \Delta t_f^{(i)} = -\tfrac{1}{2}\tau_f^{(i)}$$

Since the $\Delta t^{(i)}$ are directly related to $\tau^{(i)}$ for terminal thrusts, only one variable remains to be determined at each order, corresponding to the one interior switch point.

The first-order corrections $\tau_k^{(1)}$ are determined by the relation $\tau_k^{(1)} = \Delta V_k$. Second-order corrections $\tau_k^{(2)}$ are determined from the condition that $p(t)$ equal 1 to first order.

Terminal burns: $p(t)$ is expanded in a series in $1/a$ and in Taylor series in time for the initial and final burns. The development, shown in Appendix B, yields to order $1/a$ as follows:

$$\begin{aligned} &\text{initial burn} \quad p_1^{(1)} = -\dot{p}_1^{(0)}\tau_1^{(1)} = -\dot{p}_1^{(0)}\Delta V_1 \\ &\text{final burn} \quad p_f^{(1)} = +\dot{p}_f^{(0)}\tau_f^{(1)} = \dot{p}_f^{(0)}\Delta V_f \end{aligned} \tag{23}$$

and to order $1/a^2$,

$$p_k^{(2)} = -[\dot{p}_k^{(1)}\Delta V_k \pm \dot{p}_k^{(0)}\tau_k^{(2)} + \ddot{p}_k^{(0)}(\Delta V_k^2/2)] \tag{24}$$

In Eq. (24) the (+) applies for the initial and the (−) for the final burn. Eqs. (23) represent two conditions on $p^{(1)}$ (or, equivalently, on $\psi^{(1)}$) and Eq. (24) represents two conditions on $p^{(2)}$ (or $\psi^{(2)}$). The equations are linear in $\psi^{(1)}$ and $\psi^{(2)}$,

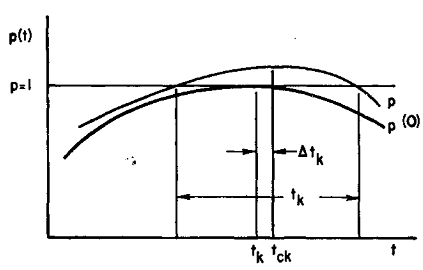

Fig. 3 Geometry of an internal burn.

respectively (see Appendix A). Note that $p_1^{(1)}$ and $p_f^{(1)}$ are both positive.

Interior burns: A typical plot of p vs t near an interior burn is shown in Fig. 3. At interior burns $\dot{p}_k^{(0)} = 0$ and $p_k^{(0)} = 1$. As in the case of terminal burns, we must have $p(t_k^+) = 1$ and $p(t_k^-) = 1$. To order $1/a$, the condition corresponding to Eq. (23) at interior burns becomes

$$p_k^{(1)} = 0 \tag{25}$$

For order $1/a^2$, Eq. (24) holds also at interior burns, with $\dot{p}_k^{(0)} = 0$.

The switch points t_k^+ and t_k^- are solutions of the equation $p(t) = 1$, expanded to the appropriate order. This equation enables us to solve for t_k and Δt_k as well. The analysis in Ref. 7 shows that the centroid movement is given by

$$\Delta t_k^{(1)} = -\dot{p}_k^{(1)}/\ddot{p}_k^{(0)} \tag{26}$$

and

$$\Delta t_k^{(2)} = -[\ddot{p}_k^{(1)}\Delta t_k^{(1)} + \tfrac{1}{2}\dddot{p}_k^{(0)}(\Delta t_k^{(1)} + \tau_k^{(1)}) + \dot{p}_k^{(2)}]/(1/\ddot{p}_k^{(0)}) \tag{27}$$

Using this information, it is now possible to determine the limits of integration which are used for evaluating the integral I_k, Eq. (21), as follows:

$$(\xi - t_k)_-^+ = \tau_k = (\Delta V_k/a) + (\tau_k^{(2)}/a^2) + (\tau_k^{(3)}/a^3) + \ldots \tag{28}$$

$$\tfrac{1}{2}(\xi - t_k)^2{}_-^+ = \Delta t_k\tau_k = (1/a^2)\Delta t_k^{(1)}\tau_k^{(1)} + (1/a^3)(\Delta t_k^{(2)}\tau_k^{(1)} + \Delta t_k^{(1)}\tau_k^{(2)}) + \ldots \tag{29}$$

$$\tfrac{1}{3}(\xi - t_k)^3{}_-^+ = (1/a^3)(\tau_k^{(1)}\Delta t_k^{(1)2} + \tfrac{1}{12}\tau_k^{(1)3}) + \ldots \tag{30}$$

In deriving these limits, it has been necessary to impose n conditions on the higher order terms of the adjoint vector (where n is the number of burns). For example, for order $1/a$, Eq. (23) represents a condition at both initial and final times, and Eqs. (25) must be applied at each interior burn. These are n linear equations in $\psi_f^{(1)}$; the $\tau_k^{(2)}$ will be used to satisfy these equations. Similarly, for order $1/a^2$, Eq. (24) represent n linear equations in $\psi_f^{(2)}$; the $\tau_k^{(3)}$ are determined by these conditions.

Evaluation of I_k

The integral in Eq. (12) must be evaluated over the entire trajectory. However, u and $u^{(0)}$ are nonzero only during the thrust intervals τ_k. Therefore Eq. (12) can be written

$$\sum_k \int_\tau \Phi(t_f,\xi)u(\xi)d\xi - \sum_k \int_{\tau_k} \Phi(t_f,\xi)u^{(0)}(\xi)d\xi + \bar{q} = 0 \tag{31}$$

where

$$\bar{q} = \int_0^{t_f} \Phi(t_f,\xi)q(\xi)d\xi \tag{32}$$

As discussed previously, $\bar{q}$ is of order $1/a^2$. Its evaluation will be discussed in the section on the second-order solution.

Using Eqs. (19) and (20), Eq. (31) can be written in the form

$$W\psi_f - W'\psi_f^{(0)} + \bar{q} = 0 \tag{33}$$

where we define

$$W = \sum_k \Phi(t_f,t_k)I_k\Phi_k^T(t_f,t_k) \tag{34}$$

and

$$W' = \sum_k \Phi(t_f,t_k)M\Phi^T(r_f,t_k)\Delta V_k \tag{35}$$

Evaluating the integral I_k, Eq. (21), using Eqs. (28–30) yields

$$I_k = I_k^{(0)} + (1/a)I_k^{(1)} + (1/a^2)I_k^{(2)} + \ldots \tag{36}$$

where

$$I_k^{(0)} = M\tau_k^{(1)}$$

$$I_k^{(1)} = M(\tau_k^{(2)} + \tau_k^{(1)}h_k^{(1)} + (M\dot{h}_k^{(0)} + N)\Delta t_k^{(1)}\tau_k^{(1)})$$

$$I_k^{(2)} = M(\tau_k^{(3)} + \tau_k^{(2)}h_k^{(1)} + \tau_k^{(1)}h_k^{(2)}) + (M\dot{h}_k^{(0)} - N)(\Delta t_k^{(2)}\tau_k^{(1)} + \Delta t_k^{(1)}\tau_k^{(2)}) + (M\dot{h}_k^{(1)} - Nh_k^{(1)})\Delta t_k^{(1)}\tau_k^{(1)} + \tfrac{1}{2}(M\ddot{h}_k^{(0)} - Nh_k^{(0)} + Q_k)[\tau_k^{(1)}\Delta t_k^{(1)2} + (\tau_k^{(1)3}/12)]$$

where the $h^{(i)}$ are defined in Appendix A. Define

$$W^{(i)} = \sum_k \Phi(t_f,t_k)I_k^{(i)}\Phi^T(t_f,t_k) \tag{37}$$

Then

$$W = W^{(0)} + (1/a)W^{(1)} + (1/a^2)W^{(2)} + \ldots \tag{38}$$

Substituting Eqs. (7) and (38) into Eq. (33) and matching terms of the same order gives

$$1/a^0\text{:}\quad W^{(0)}\psi_f^{(0)} - W'\psi_f^{(0)} = 0 \tag{39}$$

$$1/a\text{:}\quad W^{(1)}\psi_f^{(0)} \times W^{(0)}\psi_f^{(1)} = 0 \tag{40}$$

$$1/a^2\text{:}\quad W^{(2)}\psi_f^{(0)} + W^{(1)}\psi_f^{(1)} + W^{(0)}\psi_f^{(2)} = -\bar{q} \tag{41}$$

Eq. (39) is satisfied by taking $\tau_k^{(1)} = \Delta V_k$, thus making $W^{(0)} = W'$; Eqs. (40) and (41) are to be solved for $\psi_f^{(1)}$ and $\psi_f^{(2)}$, respectively. The "forcing term" on the right-hand-side of Eq. (41) is known once Eq. (40) is solved. This procedure can, in principle, be carried out to any order, although the forcing terms become algebraically intractable for higher order than second.

First-Order Solution

The equation which determines the first-order correction $\psi^{(1)}$ is

$$W^{(1)}\psi_f^{(0)} + W^{(0)}\psi_f^{(1)} = 0 \tag{42}$$

This cannot be solved by simply inverting $W^{(0)}$ since $W^{(1)}$ depends on $\psi_f^{(1)}$ and on $\tau_k^{(2)}$ (which as yet are unknown). The product $W^{(1)}\psi_f^{(0)}$ can be written

$$W^{(1)}\psi_f^{(0)} = A^{(1)}\psi_f^{(1)} + B^{(1)}\psi_f^{(0)} + C^{(1)}\psi_f^{(0)} \tag{43}$$

where

$$A^{(1)} = -\sum_{k=2}^{n-1} \frac{\Delta V_k}{\ddot{p}_k^{(0)}} e_k e_k^T$$

$$B^{(1)} = -\sum_{k=1,n} \Delta t_k^{(1)}\Delta V_k\Phi_k(M\dot{p}_k^{(0)} - N)\Phi_k^T$$

$$C^{(1)} = \sum_k \tau_k^{(2)}\Phi_k M\Phi_k$$

and

$$e_k = \Phi_k N\psi_k^{(0)} \text{ and } \Phi_k = \Phi(t_f,t_k)$$

Note that the summation in $A^{(1)}$ is only over interior thrusts and the summation in $B^{(1)}$ is only over terminal thrusts.

The formal solution to Eq. (42) is then

$$\psi_f^{(1)} = -[W^{(0)} + A^{(1)}]^{-1}[B^{(1)} + C^{(1)}]\psi_f^{(0)} \tag{44}$$

By its definition [see Eq. (37)] $W^{(0)}$ is a positive semi-definite symmetric matrix; it can be shown[7] that it must be nonsingular (i.e., positive definite) if the optimal control is unique. $A^{(1)}$ is also positive semidefinite (note: $\ddot{p}^{(0)} < 0$ at interior

Table 1 Test case I: second order

Boundary conditions	r_1	r_2	r_3	v_1	v_2	v_3
Initial	1.00000	0.00000	0.0	0.00000	1.000000	0.0
Final	−1.06066	1.06066	0.0	−0.57735	−0.57735	0.0
Impulsive adjoint	λ_1	λ_2	λ_3	$\dot{\lambda}_1$	$\dot{\lambda}_2$	$\dot{\lambda}_3$
Initial	0.68563	0.72795	0.0	−1.01023	−0.53969	0.0
Final	−0.15900	−0.98727	0.0	0.09224	−0.68069	0.0
Impulsive ΔV				Trip time		
$\Delta V_1 = 0.1271156$ $\Delta V_2 = 0.1087456$				$t_f = 3.3028$		
First-order terms				Second-order terms		
$\tau_1^{(2)} = 0.00854$ $\tau_2^{(2)} = 0.00411$ $\psi_1^{(1)} = \begin{bmatrix} 0.1326 \\ 0.0937 \\ 0.0 \\ 0.1289 \\ 0.0682 \\ 0.0 \end{bmatrix}$				$\tau_1^{(3)} = 0.00102$ $\tau_2^{(3)} = 0.00036$ $\psi_1^{(2)} = \begin{bmatrix} 0.0278 \\ 0.0118 \\ 0.0 \\ 0.0180 \\ 0.0205 \\ 0.0 \end{bmatrix}$		

impulses). Therefore, $(W^{(0)} + A)^{-1}$ exists if the optimal impulsive control is unique.

The one complication in the solution (44) is that $C^{(1)}$ is not yet known, since it depends upon $\tau_k^{(2)}$. This is resolved in the following manner. The third term in Eq. (43) can be written $C^{(1)}\psi_f^{(0)} = Uc^{(1)}$ where $U = [\Phi_1 M\psi_1|\Phi_2 M\psi_2|\ldots|M\psi_f]$ and $c^{(1)} = \text{col}(\tau_1^{(2)}\ldots\tau_f^{(2)})$. In addition, we can define the vector

$$\ell^{(1)} = \begin{pmatrix} p_1^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ p_f^{(1)} \end{pmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \psi_k^{(0)T} M \psi_k^{(1)} \\ \cdot \\ \cdot \end{bmatrix} = U^T \psi_f^{(1)}$$

From the conditions imposed in the switch point analysis, Eqs. (23) and (25), $\ell^{(1)}$ is known.

Multiplying Eq. (44) by U^T and rearranging

$$U^T(W^{(0)} + A^{(1)})^{-1}Uc^{(1)} = b^{(1)} - l^{(1)} \tag{45}$$

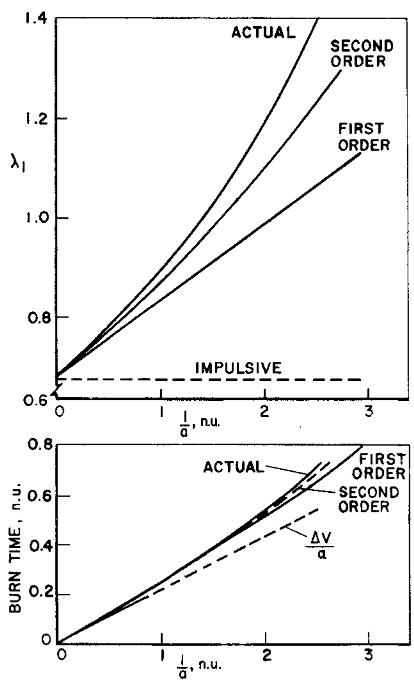

Fig. 4 Results for case I, Earth-Mars transfer, 192 days.

where

$$b^{(1)} = -U^T(W^{(0)} + A^{(1)})^{-1}B^{(1)}\psi_f^{(0)}$$

is a known vector.

In Ref. 8 Potter and Stern show that the columns of U are linearly independent if the optimal trajectory has the minimum number of impulses. Assuming this to be the case, the matrix on the left-hand-side of Eq. (45) is invertible and

$$c^{(1)} = [U^T(W^{(0)} + A^{(1)})^{-1}U]^{-1}(b^{(1)} - l^{(1)}) \tag{46}$$

Therefore, the $\tau_k^{(2)}$ are determined from Eq. (46) and then $\psi_f^{(1)}$ is determined from Eq. (44). The above criteria on the invertibility of the matrices $[W^{(0)} + A^{(1)}]$ and $[U^T(W^{(0)} + A^{(1)})^{-1}U]$ are sufficient conditions for an impulsive trajectory to be a limit case of a family of finite-thrust trajectories satisfying the same boundary conditions.[7]

Second-Order Solution

The equations to be solved for the second-order terms $\psi_f^{(2)}$ are

$$W^{(0)}\psi_f^{(2)} + W^{(1)}\psi_f^{(1)} + W^{(2)}\psi_f^{(0)} = -\bar{q} \tag{47}$$

The same complications arise as in the first order case; $W^{(2)}$ is a (linear) function of $\psi_f^{(2)}$ and $\tau_k^{(3)}$. Again $W^{(2)}\psi_f^{(0)}$ can be written

$$W^{(2)}\psi_f^{(0)} = A^{(2)}\psi_f^{(2)} + B^{(2)}\psi_f^{(0)} + C^{(2)}\psi_f^{(0)} \tag{48}$$

where

$$A^{(2)} = A^{(1)}$$

$$B^{(2)} = \sum_k \Phi_k[M(\tau_k^{(2)}h_k^{(1)}) + (M\dot{h}_k^{(0)} - N)(\Delta t_k^{(2)}\tau_k^{(1)} + \Delta t_k^{(1)}\tau_k^{(2)}) + (M\dot{h}_k^{(1)} - Nh_k^{(1)}\Delta t_k^{(1)}\tau_k^{(1)}) + \tfrac{1}{2}(M\ddot{h}_k^{(0)} - N\dot{h}_k^{(0)} + Q_k)\tau_k^{(1)}\Delta t_k^{(1)2} + (\tau_k^{(1)3}/12)]\Phi_k^T$$

$$C^{(2)} = \sum_k \tau_k^{(3)}\Phi_k M \Phi_k^T$$

Proceeding as before, the formal solution to Eq. (47) is

$$\psi_f^{(2)} = -(W^{(0)} + A^{(1)})^{-1}\xi \tag{49}$$

where

$$\xi = (B^{(2)} + C^{(2)})\psi_f^{(0)} + W^{(1)}\psi_f^{(1)} + \bar{q}$$

Again it is possible to write $C^{(2)}\psi_f^{(0)} = Uc^{(2)}$ where now

Table 2 Test case II: second order

Boundary conditions	r_1	r_2	r_3	v_1	v_2	v_3
Initial	1.00000	0.00000	0.0	0.00000	1.00000	0.0
Final	−3.5355	3.5355	0.0	−0.31623	−0.31623	0.0
Impulsive adjoint	λ_1	λ_2	λ_3	$\dot{\lambda}_1$	$\dot{\lambda}_2$	$\dot{\lambda}_3$
Initial	0.7437	0.6685	0.0	−0.8027	−0.4047	0.0
Final	−0.2086	−0.9780	0.0	−0.0289	−0.1357	0.0
Impulsive ΔV				Trip time		
$\Delta V_1 = 0.39289$ $\Delta V_2 = 0.2302$				$t_f = 12.0000$		
First-order terms				Second-order terms		
$\tau_1^{(2)} = 0.0649$ $\tau_2^{(2)} = 0.0057$ $\psi_1^{(1)} = \begin{bmatrix} 0.2426 \\ 0.1833 \\ 0.0 \\ 0.3132 \\ 0.1614 \\ 0.0 \end{bmatrix}$				$\tau_1^{(3)} = 0.0149$ $\tau_2^{(3)} = 0.0012$ $\psi_1^{(2)} = \begin{bmatrix} 0.1098 \\ 0.0228 \\ 0.0 \\ 0.0622 \\ 0.1235 \\ 0.0 \end{bmatrix}$		

$c^{(2)} = \text{col}(\tau_1^{(3)} \ldots \tau_f^{(3)})$. The vector $l^{(2)} = U^T\psi_f^{(2)}$ is known from switch point analysis, combining Eq. (24) with the expression for $p^{(2)}$ from Appendix A. Therefore, multiplying Eq. (49) by U^T and solving for $c^{(2)}$,

$$c^{(2)} = [U^T(W^{(0)} + A^{(1)})^{-1}U]^{-1}(b^{(2)} - l^{(2)}) \qquad (50)$$

where

$$b^{(2)} = U^T(W^{(0)} + A^{(1)})^{-1}(B^{(2)}\psi_f^{(0)} + W^{(1)}\psi_f^{(1)} + \bar{q})$$

As in the first order case Eq. (50) is solved for $\tau_k^{(3)}$ and then Eq. (49) for $\psi_f^{(2)}$.

As yet, we have not discussed the evaluation of $\bar{q}$. This is done in Appendix C. For an inverse square field, $\bar{q}$ can be evaluated analytically without integration.

For cases in which the impulses are small so that the resulting trajectory does not deviate greatly from a coast arc, the $\bar{q}$ term becomes negligible. This is the second-order linear case and our limited numerical experience indicates that results which are sufficiently accurate for mission analysis can be obtained while ignoring $\bar{q}$. The effort required to program the evaluation of $\bar{q}$ is, therefore, probably not justified, in general (although once programmed, execution time is negligible).

Convergence Properties

The solution developed here is in the form of a series in $1/a$. Each of the resulting terms of the expansion is, however, a function of ΔV_k. In particular, $W^{(i)} = O(\Delta V^{i+1})$ $\tau^{(i)} = O(\Delta V^i)$ and $\psi^{(i)} = O(\Delta V^1)$. An analysis of the terms appearing in the series indicates that the parameter $\Delta V/a$ (which is also the ratio of burn time to the characteristic time of the problem) determines the rate of convergence. We can expect, therefore, that this solution will be valid for trajectories whose thrust periods are short compared to the total trip time. Numerical experience thus far indicates that this often includes thrust levels appropriate to the electric propulsion category.

It can also be shown that $q = O(\Delta V^4)$. Thus, if ΔV is "small," which is generally the case, $\bar{q}$ is negligible and the linear form of the second-order solution is valid.

Numerical Results

The method developed in this paper was applied to three representative coplanar heliocentric space trajectories: 1) a 192-day Earth-Mars rendezvous, 2) a 698-day Earth-Jupiter rendezvous, and 3) a 259-day Earth-Mars rendezvous. The trajectory data are summarized in Tables 1, 2, and 3.‡ In Figs. 4–6 the trajectory data are plotted versus $1/a$. The top half of each figure shows a plot of one component of the primer vector vs $1/a$. Other components behave similarly. The unit of acceleration for this plot is AU/TAU² or about 0.006 m/sec.² For the impulsive case ($1/a = 0$), the actual and computed values are the same. The first-order correction is a straight line, with the correct slope from this point. The second-order correction is a parabola, with the correct both slope and curvature at $1/a = 0$. The estimate which would be used in the impulsive iterative method is indicated in the figures by a dashed line. Note that the analytic method gives results within 5% for acceleration levels as low as 0.003

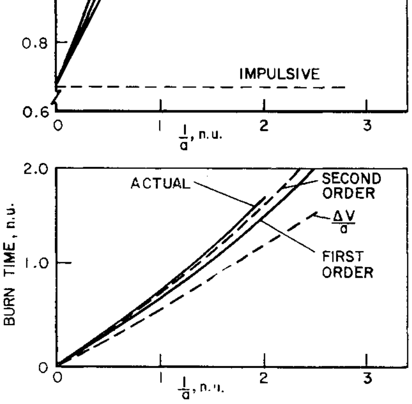

Fig. 5 Results for case II, Earth-Jupiter transfer, 697.6 days.

‡ A normalized unit system is used for computational efficiency. The unit of distance is the astronomical unit (AU), the unit of velocity is Earth's mean orbital speed (EMOS), and the unit of time (TAU) is the time for Earth to go one radian (about 60 days).

Table 3 Test case III: three-burn trajectory

Boundary conditions	r_1	r_2	r_3	v_1	v_2	v_3
Initial	1.00000	0.00000	0.0	0.00000	1.00000	0.0
Final	−1.1769	−0.9676	0.0	0.5145	−0.6258	0.0
Position of midcourse burn	0.3939	0.8900	0.0	(−0.9435)	0.4008 (before the impulse)	0.0
Impulsive adjoint	λ_1	λ_2	λ_3	$\dot{\lambda}_1$	$\dot{\lambda}_2$	$\dot{\lambda}_3$
Initial	0.9959	−0.0903	0.0	0.6781	0.7164	0.0
Midcourse	−0.9793	0.2024	0.0	−0.0768	−0.3715	0.0
Final	0.9929	0.1188	0.0	0.8974	0.3946	0.0
Impulsive ΔV:				Trip time		
$\Delta V_1 = 0.0269$ $\Delta V_2 = 0.1100$ ($t_2 = 1.1235$) $\Delta V_3 = 0.1610$				$t_f = 4.4531$		
First-order terms						
$\tau_1^{(2)} = 0.0100$ $\tau_2^{(2)} = -0.0019$ $\tau_3^{(2)} = 0.0043$ $\Delta t_2^{(1)} = -0.0625$				$\psi_1^{(1)} = \begin{bmatrix} -0.0237 \\ -0.0045 \\ 0.0 \\ -0.0231 \\ -0.0344 \\ 0.0 \end{bmatrix}$		

m/sec.[2] In the bottom half of each figure, the total burn time is plotted vs $1/a$. Again the actual integrated values can be compared with the first- and second-order analytical computation. The straight line at the bottom represents a "zeroth order" calculation: the total ΔV divided by a. In this case, the results are accurate for still lower values of a.

The value of the analytic method for predicting initial guesses for the adjoint variables is shown by the example of the 697.6-day Earth-Jupiter case for a value of $a = 1$ (normalized units). After 25 iterations the program failed to converge using the impulsive adjoint. Convergence is obtained, however, in 13 iterations with first-order corrections included and in only 7 iterations with second-order corrections. The running times were 6 min 9 sec, 2 min 18 sec, and 1 min 17 sec, respectively. The additional computing time to perform the analytic computations necessary to make the first- and second-order corrections is only a fraction of one second on an IBM 360/65 computer.

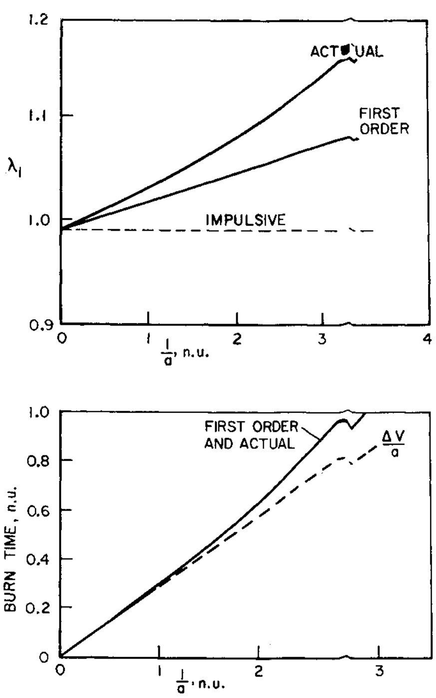

Fig. 6 Results for case III, Earth-Mars Transfer, 258.9 days.

Summary

A method for calculating estimates of the adjoint vector for optimal space trajectories based on the impulsive case has been developed. The method results in significant reductions in computing time. Calculation of the estimate takes less than one second of IBM 360/65 time.

This method is also capable of accurate predictions of finite-thrust burn times, and therefore, provides a useful tool for performing vehicle performance computations with almost negligible computation time.

Appendix A

The primer magnitude p can be written

$$p^2 = (\lambda \cdot \lambda) \tag{A1}$$

Expanding the primer vector $\lambda = \lambda^{(0)} + (1/a)\lambda^{(1)} + \ldots$ and substituting in Eq. (A1),

$$p^2 = [\lambda^{(0)} + (1/a)\lambda^{(1)} + \ldots][\lambda^{(0)} + (1/a)\lambda^{(1)} + \ldots]$$

Therefore

$$p = p^{(0)} + (1/a)p^{(1)} + (1/a^2)p^{(2)} + \ldots$$

where

$$p^{(0)} = (\lambda^{(0)} \cdot \lambda^{(0)})^{1/2}, \quad p^{(1)} = \lambda^{(0)} \cdot \lambda^{(1)}/p^{(0)}$$

$$p^{(2)} = \frac{\lambda^{(0)} \cdot \lambda^{(2)}}{p^{(0)}} + \frac{1}{2p^{(0)}}\left[\lambda^{(1)} \cdot \lambda^{(1)} - \left(\frac{\lambda^{(0)} \cdot \lambda^{(1)}}{p^{(0)}}\right)^2\right]$$

Similarly

$$\dot{p} = \dot{p}^{(0)} + (1/a)\dot{p}^{(1)} + (1/a^2)\dot{p}^{(2)} + \ldots$$

where $\dot{p}^{(1)} = (d/dt)p^{(1)}$. Expanding $p(t)$ in a Taylor series about $p(t_k) = 1$,

$$p(t) = 1 + \dot{p}_k(t - t_k) + \ddot{p}_k[(t - t_k)^2/2]$$

Inverting this series using the fact that $(t - t_k)$ is small,

$$h(t) = 1/p(t) = h_k + \dot{h}_k(t - t_k) + \ddot{h}_k[(t - t_k)^2/2] + \ldots$$

where

$$h_k = 1 - (1/a)p_k^{(1)} + (1/a^2)(p_k^{(1)2} - p_k^{(2)}) + \dots$$

$$\dot{h}_k = -\dot{p}_k^{(0)} + (1/a)(2p_k^{(1)}.\dot{p}_k^{(0)} - \dot{p}_k^{(1)}) + \dots$$

$$h_k = 2\dot{p}_k^{(0)2} - \ddot{p}_k^{(0)} + \dots$$

For brevity this is also written in the form

$$\dot{h}_k = h_k^{(0)} + (1/a)h_k^{(1)} + (1/a^2)h_k^{(2)} + \dots$$

Appendix B

Consider the switch equation for the initial burn $p(\tau_1) = 1$. Expanding in Taylor series,

$$p_1 + \dot{p}_1\tau_1 + \ddot{p}_1(\tau_1^2/2) = 1$$

Substituting the expansion in $1/a$ (with $p_1^{(0)} = 1$),

$$p_1 = 1 + (1/a)p_1^{(1)} + (1/a^2)p_1^{(2)} + \dots$$

$$\dot{p}_1 = \dot{p}_1^{(0)} + \frac{1}{a}\dot{p}_1^{(1)} + \dots, \ddot{p}_1 = \ddot{p}_1^{(0)} + \dots$$

and the expansion for τ_1,

$$\tau_1 = (\tau_1^{(1)}/a) + (\tau_1^{(2)}/a^2) + \dots$$

and equating terms of like order,

$$1/a: p_1^{(1)} + \dot{p}_1^{(0)}\tau_1^{(1)} = 0$$

$$1/a^2: \; p_1^{(2)} + \dot{p}_1^{(1)}\tau_1^{(1)} + \dot{p}_1^{(0)}\tau_1^{(2)} + \ddot{p}_1^{(0)}\tau_1^{(1)2}/2 = 0$$

An analogous procedure is followed for other burns.

Appendix C

In this appendix, an explicit expression for $\bar{q}$ [Eq. (47)] is derived. The term $\bar{q}$ appears in the second-order equations because the equations of motion are nonlinear [Eq. (1)]. By definition

$$q = \tfrac{1}{2}\Delta x^T(\partial^2 f/\partial x^2)\partial x, \quad = \tfrac{1}{2}\Delta r^T(\partial^2 g/\partial r^2)\Delta r$$

Since Δr is always of order $1/a$, q is always of order $1/a^2$.[7] The second-order terms of q are given by

$$q = \tfrac{1}{2}x^{(1)T}(\partial^2 f/\partial x^2)x^{(1)}$$

The vector $\bar{q}$ is defined as

$$\bar{q} = \int_0^{t_f} \Phi(t_f,\tau)q(\tau)d\tau \tag{C1}$$

The elements of $\bar{q}$ can be rewritten

$$q_i = \tfrac{1}{2}\int_0^{t_f} x^{(1)T}H_{xx}^{(i)}x^{(1)}d\tau$$

where the matrix

$$H_{xx}^{(k)} = (\partial^2/\partial x^2)(\psi^{(i)}\cdot f)$$

and $\psi^{(k)}$ is the kth row of $\Phi(t_f,\xi)$.

This expression can be further simplified using the relation $x^{(1)}(t) = \Phi(t,t_k^+)x\, x^{(1)}(t_k^+)$:

$$q_i = \tfrac{1}{2}\sum_{k=1}^{n} \Delta x_k^{(1)T}Q_k^{(i)}\Delta x_k^{(1)} \tag{C2}$$

where

$$Q_k^{(i)} = \int_{t_k}^{t_{k+1}} \Phi^T(t,t_k)H_{xx}^{(i)}\Phi(t,t_k)dt \tag{C3}$$

The form (C2) assumes that $x^{(1)}(0) = 0$.

The vector $x^{(1)}$ is the difference to the first order between impulsive and finite-thrust trajectories. From Eqs. (19–21) and (37) the change in $x^{(1)}$ over an impulse is

$$\Delta x(t_k) = I_k^{(1)}\psi_k = M(\tau_k^{(2)} + \tau_k^{(1)}h_k^{(1)}) + (M\dot{h}_k^{(0)} + N)\Delta t_k^{(1)}\tau_k^{(1)}$$

Note that there is a position displacement as well as a velocity impulse since N appears.

It is also shown in Ref. 7 that for an inverse square gravitational field $Q^{(i)}$ can be evaluated analytically. Furthermore, the error resulting from ignoring the integral (C1) over the thrust interval is of order $1/a^3$. A similar analysis shows that the integral (C3) can be taken from t_k to t_{k+1} instead of t_k^+ to t_{k+1}^- with an error of order $1/a^3$.

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